Complex dynamics. Hyperbolic Julia sets.

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Let $Q(z)$ be a polynomial of degree $d\geq 2$. For simplicity of notations, assume $Q(z)$ is monic, i.e. $Q(z)=z^d+R(z)$, $\deg Q\leq d-1$

The basin of attraction of ∞, $\Omega_\infty(Q):=\{z\in\mathbb{C}: Q^{\circ n}(z)\to\infty\}$. Contains $(B(0,R))^c$ for large R.

The Julia set of Q is defined as $J(Q)=\partial\Omega_\infty(Q)$ – the "chaotic set".

Lemma $J(Q)$ is a non-empty compact.

Pf. Let $B=\{|z|>R\}\subset\Omega_\infty(P)$, $B\supset P(B)$. Then $\Omega_\infty(Q)=\bigcup Q^{-n}(B)$ (every point escaping to ∞ eventually enters B).

$\Omega_\infty(Q)^c=\bigcap(Q^{-n}(B))^c$, each $(Q^{-n}(B))^c$ is a compact set, $(Q^{-(n+1)}(B))^c\subset(Q^{-n}(B))^c$ (since $Q(B)\supset B$). Thus $K(P):=\bigcap(Q^{-n}(B))^c\neq\emptyset$ – filled-in Julia set.

So is its boundary $\partial K(Q)=\partial\Omega_\infty(Q)=J(Q)$.

Since $\Omega_\infty(Q)$ is completely invariant ($P(\Omega_\infty(Q))=P^{-1}(\Omega_\infty(Q))=\Omega_\infty(Q)$), so is $J(Q)$ and $K(Q)$.

Define now Bötcher coordinates in $\Omega_\infty(Q)$. In the following way:

In B $(=\{|z|>R\},\ B\subset Q(B))$, define

$\varphi_n(z)=(Q^{\circ n}(z))^{d^{-n}}=z(1+\dots)^{d^{-n}}$.

This is well defined, since in B, $|Q^{\circ n}(z)|>|z|$, i.e. $|(1+\dots)|>1$, so $\neq 0$.

Observe that $\varphi_n(z)^d=\varphi_{n-1}(Q(z))$. Now $\frac{\varphi_{n+1}(z)}{\varphi_n(z)}=\left(\frac{Q^{\circ n+1}(z)}{(Q^{\circ n}(z))^d}\right)^{d^{-(n+1)}}=\left(1+\frac{R(Q^{\circ n}(z))}{(Q^{\circ n}(z))^d}\right)^{d^{-(n+1)}}\leq\left(1+\frac{1}{Q^{\circ n}(z)}\right)^{d^{-(n+1)}}$ for large $|z|$.

Thus $\lim_{n\to\infty}\varphi_n(z)=:\varphi_P(z)=\varphi_0(z)\prod_{n=0}\frac{\varphi_{n+1}(z)}{\varphi_n(z)}$ exists in B and satisfies $\varphi_P(P(z))=\varphi_P(z)^d$.

Use this relation to inductively define φ_P in $\bigcup Q^{-n}(B)=\Omega_\infty(Q)$.

If $J(Q)$ is connected, $\varphi_Q(\Omega_\infty(Q))\to\{|W|>1\}$ – conformal, conjugates Q to z^d.

If $J(Q)$ is not connected, i.e. $\Omega(Q)$ is not simply-connected, then $\varphi_Q(z)$ is a multivalued function. Yet $G_Q(z)=\log|\varphi_Q(z)|$ is a well-defined function $G_Q(z)=\lim_{n\to\infty}\frac{1}{d^n}\log|Q^{\circ n}(z)|$ is the escape rate of z. It is harmonic in $\Omega_\infty(Q)$ (as a log of a real part of analytic function), has log singularity at ∞, and $G_Q(z)\to 0$ as $z\to J(Q)$. For an arbitrary compact $J\subset\mathbb{C}$, such a function is called Green function at ∞.

Define the extended Green function by $\tilde{G}_Q(z)=\begin{cases}G_Q(z), & z\in\Omega_\infty\\ 0, & z\in K(Q)\end{cases}$.

$\tilde{G}_Q(z)$ is continuous, harmonic for $z\notin J(Q)$, and subharmonic for $z\in J(Q)$.

(since $\tilde{G}_Q(z)=0\leq\frac{1}{2}\int_0^{2\pi}\tilde{G}_Q(z+re^{i\theta})\,d\theta\ \forall r$)

Then $-\frac{1}{2\pi}\Delta\tilde{G}_P(z)$ (in the sense of distributions) is a positive measure on $J(Q)$.

By Green's thm, it is a probability measure. It is called harmonic measure of $\Omega_\infty(Q)$ evaluated at ∞, ω_Q

(The same construction produces harmonic measure for any $J\subset\mathbb{C}$ – compact, and any z in the component of ∞ of $\mathbb{C}\setminus J$)

Notice that since $G_Q(Q(z))=d\,G_Q(z)$, we have $\omega_Q(Q(K))=d\,\omega_Q(K)$ for $K\subset J$, such that $Q|_K$ is injective. Thus, ω_Q is Q-invariant, moreover, $J_\omega=d$.

Since Q is a d-to-1 map, ω is thus measure of maximal entropy $\log d$.

Some classical definitions of harmonic measure:

1) Ω – a domain in $\mathbb{R}^n$, $f\in C(\partial\Omega)$. u – solution of the Dirichlet problem: the unique u: $\Delta u=0$ in Ω, $u=f$ on $\partial\Omega$ (except possibly a set of $(d-2)$-capacity 0). Fix z_0 $f\to u(z_0)$ – bounded (by the Maximum principle) linear functional, so $\exists!\ \omega_{z_0}$ – harmonic measure: $u(z_0)=\int f\,d\omega_{z_0}$ $\forall f$. $z\to\omega_z$ depends harmonically on z,

$\forall K \subset \partial\Omega$, $\omega_z(K)$ – harmonic in Ω, $=1$ on K, $=0$ on $\partial\Omega \setminus K$ (except for $\mathrm{Cap}_{d-2} = 0$).

2) $\forall$ any space with Brownian motion/Random walk: X, $\partial X \subset X$ – boundary, $z_0 \in X$: $T = \inf\{t : B_t^{z_0} \in \partial X\}$, where $B_t^{z_0}$ – BM/RW started at z_0, $B_T^{z_0}$ – random pt. of ∂X. Defining $\omega_{z_0}(K) = P(B_T^{z_0} \in K)$.

Thm (Kakutani). Harmonic measure for d-dimensional BM is harmonic measure from def. 1.

3) Let Ω be a simply-connected domain, $\varphi : \mathbb{D} \to \Omega$ – Rieman map, $\varphi(0) = z_0 \in \Omega$. Then $\omega_{z_0} = \varphi_* m$, ($m$ – normalised linear measure on $\mathbb{T}$) well-defined, since $\lim_{r \to 1^-} \varphi(r\xi)$ $\exists$ a.e. (even quasi-everywhere) $\xi \in \mathbb{T}$.

Now let us return to Complex Dynamics.

Before moving further, we need

Def. A point x is called *periodic* of period n if $Q^{\circ n}(x) = x$.
A periodic x is *repelling* if $|(Q^{\circ n})'(x)| > 1$.
attracting if $|(Q^{\circ n})'(x)| < 1$.
neutral if $|(Q^{\circ n})'(x)| = 1$.

If x is attracting, $x \notin J(Q)$ (no point from a nbhd of x escape to ∞).
If x is repelling, then $x \in J(Q)$. Indeed, if not, then $\exists U \ni x$-nbhd, s.t. $Q^k|_U$ is uniformly bounded! Thus $\exists n_k$: $f^{n_k} \to g$ uniformly on compacts. But $g'(x) = \lim f_{n_k}'(x) = \infty$ – contradiction! – to simplify notations

Finally, if x is neutral, then $x \notin J(Q) \iff \exists \varphi : \varphi(Q(z) - x)) = \lambda(z - x)$ nbhd of x.

Thm (Fatou). #attracting + neutral periodic pts $\le 6d - 6$.

Lemma. $J(Q) = \mathrm{Clos}\{\text{repelling periodic pts}\}$.

Pf We'll use Montel's Thm: let g_n be a sequence of analytic functions on open U which omits two points, i.e. $\exists w_1, w_2$: $g_n(z) \ne w_i$ $\forall z \in U$. Then $\exists$ $g_{n_k}(z) \to g(z)$ uniformly on compact subsets of U. ($g(z)$ can be $\equiv \infty$).

Assume now that $U \cap J$ – open, U – does not contain fixed points. By possibly decreasing U, can assume that there are well defined branches of P^{-1} on U. Let f_1, f_2 be two such branches. Then
$g_n(z) = \dfrac{Q^{\circ n} - f_1}{Q^{\circ n} - f_2}$ does not take value 0 and 1 (since $P^{\circ n} \ne f_1, f_2$ in U, $f_1(z) \ne f_2(z)$))
So $\exists$ $g_{n_k} \to g$ uniformly on compacts in U. So $Q^{n_k} = \dfrac{f_1 - f_2}{1 - g_{n_k}} \to \dfrac{f_1 - f_2}{1 - g}$.
For $z \in \Omega_\infty(Q) \cap U$, $Q^{\circ n}(z) \to \infty$, so $g \equiv 1$, so $\forall z \in U$, $Q^{\circ n_k}(z) \to \infty$ – contradiction! So $U \cap J(P)$ contains a periodic point. Since there are only finitely many neutral periodic points, repelling points are dense in $J(Q)$ ∎

Lemma Q is topologically-mixing on $J(Q)$, i.e. $\forall U$ $\exists n$: $Q^{\circ n}(U) \supset J(Q)$

Pf $U \cap J(Q)$ contains a repelling periodic x and a nbhd $x \in V \subset U$, such that $Q^n(V) \supset V$. Then $Q^{\circ nk}(V) \subset Q^{\circ n(k+1)}(V)$, and $Q^{\circ n(k+1)}(V) \supset Q^{\circ nk}(V)$.
Assume now that $\exists z_1 \ne z_2$, $z_1, z_2 \notin P^{\circ nk}(V)$. Then $\exists k_\ell$ $Q^{\circ nk_\ell}(z)$ converges $\forall z \in V$, uniformly on compacts. Then for $z \in \Omega_\infty(Q) \cap V$ ($\ne \emptyset$, since $V \cap J(Q) \ne \emptyset$, V – open), $Q^{\circ nk_\ell}(z) \to \infty$, for $z \in J(P) \cap V$, $Q^{\circ nk}(z)$ is bounded – contradiction.
Thus $\exists$ at most one z_0 with $z_0 \notin Q^{\circ nk}(V)$ $\forall k$. Then z_0 is $Q^{-n}(z_0)$, thus $Q^{-n}(z_0) = z_0$. So $Q^n(z) = (z - z_0)^d + z_0$, and $Q^{\circ nk}(z) \to z_0$ in a nbhd of z_0 ($P'(z_0) = 0$!) so $z_0 \notin J$ ∎

Now, let us consider a case of *hyperbolic Julia sets.*

Def. Let $C(Q) = \mathrm{Clos}\Big(\bigcup_{P'(c)=0} Q^{\circ n}(c)\Big)$ – the *postcritical set* of Q.
We say that Q is *hyperbolic* if $C(Q) \cap J(Q) = \emptyset$.

Lemma. If Q is hyperbolic [illegible] we say that Q is hyperbolic if $C(Q)\cap J(Q)=\emptyset$.

Lemma. If Q is hyperbolic, then $(J(Q), V, Q)$ is CER for some neighborhood V of $J(Q)$.

Pf. Let $U := \mathbb{C}\setminus C(Q)$ – open, $J(Q)\subset U$. $Q^{-1}(U)\subset U$.

Then $Q: Q^{-1}(U)\to U$ is a local cover (no critical points!), d-to-1. By the uniformization Thm, $\exists$ covering map $\tau:\mathbb{D}\to U$

(there is one exception: $C(P)=\{z_0\}$, but in this case again $P(z)=(z-z_0)^d+z_0$, so $J(P)=\{|z-z_0|=1\}$, and the Thm is clear.)

$$\begin{tikzcd} \mathbb{D} \arrow[r, "\xi"] \arrow[d, "\tau"] & \mathbb{D} \arrow[d, "\tau"] \\ U \arrow[r, "Q"] & U \end{tikzcd}$$

Since P is a cover, there is a lift $\xi:\mathbb{D}\to\mathbb{D}$, such that $\forall z\in\mathbb{D}$, $\tau(z)=Q(\tau(\xi(z)))$.

(ξ is a lift of P^{-1}). Since J contained a repelling periodic point, ξ is a strict contraction in $\rho_{\mathbb{D}}$ – Hyperbolic metric on $\mathbb{D}$.

Define now the hyperbolic metric on U by $\tau_U := \eta_U(z)dz$, where $\eta_U(z) := \dfrac{\eta_{\mathbb{D}}(\tau^{-1}(z))}{|\tau'(\tau^{-1}(z))|}$ – well defined since $\eta_{\mathbb{D}}$ is automorphism – invariant.

Thus, for some $\lambda>1$, $z_1, z_2\in J$ we have $\rho_U(P(z_1),P(z_2))\geq\lambda\rho_U(z_1,z_2)$, and $C>\eta_U(z)>C^{-1}$ for $z\in J$. Then let $z_1\to z_2$, to get

$$|(Q^{\circ n})'(x)|\geq\lambda^n C^2,$$ same as for Cantor sets.

Now we need to find V. If W – bdd. nbhd of $J(Q)$: $\dfrac{\rho_U(Q(z_1),Q(z_2))}{\rho_U(z_1,z_2)}>\lambda$ $\forall z_1,z_2\in W$. If $N_\varepsilon=\{z\in W:\ \rho_U(z,J_c)<\varepsilon\}$ with ε so small that $N_\varepsilon\cap Q(\hat{\mathbb{C}}\setminus W)=\emptyset$. Then $Q^{-1}(N_\varepsilon)\subset W$, so $Q^{-1}(N_\varepsilon)\subset N_{\varepsilon/2}\subset N_{\frac{\varepsilon}{\lambda}}\subset N_\varepsilon$.

so $V=N_\varepsilon$ works ∎

For hyperbolic Julia sets, ω is the measure of maximal entropy, so $P(0)=\log d$, and thus we have

$$\operatorname{Hdim}\{x\in J:\ \lim\frac{\log(\omega(B(x,\varepsilon)))}{\log\varepsilon}=\alpha\}=\inf\left(t+\alpha\frac{P(t)}{\log d}\right).$$

For hyperbolic Julia sets, there are more intrinsic definitions of $P(t)$:

1) $P(t)=\lim\limits_{n\to\infty}\frac{1}{n}\log\sum\limits_{Q^{\circ n}(y)=x}|(Q^{\circ n})'(y)|^{-t}$ for any $x\in J$ – since $(Q^{-n})(x)$ is dense in J, and visits every cylinder.

2) $P(t)=\lim\limits_{n\to\infty}\frac{1}{n}\log\sum\limits_{Q^{\circ n}(x)=x}|(Q^{\circ n})'(x)|^{-t}$.

Also, we can prove that

Thm. 1) $\dim\omega\leq 1$.

2) TFAE

a) $J(Q)$ is connected.

b) $\forall c:\ Q'(c)=0\Rightarrow Q^{\circ n}(c)$ is bounded.

c) $\dim\omega=1$.

Pf. As we know, $\dim\omega=\dfrac{h_\omega}{\lambda_\omega}$. $h_\omega=\log d$.

On the other hand, $P'(z)=d\prod(z-c_j)$, $P'(c_j)=0$, so

$\lambda_\omega=\int\log|P'|d\omega=\log d+\sum\limits_{j=1}^{d-1}\log|z-c_j|\,d\omega(z)=$ (integration by parts) $\log d+\sum\limits_{j=1}\int\log|z-c_j|\Delta G_p\,dA=\log d+\sum G_p(c_j)$. Thus $\lambda_\omega\geq\log d$, with equality iff $G_p(c_j)=0\ \forall j$, so $c_j\notin\Omega_\infty(Q)$.

If $c_j\notin\Omega_\infty(Q)$, then $\Omega_\infty(Q)=\bigcup P^{-n}(B)$ is a union of simply-connected domains, so it is simply-connected, and $J(P)$ is connected.

If $J(Q)$ is connected, then $\Omega_\infty(Q)$ – simply connected, and $\Phi(z)$ conjugates Q to z^d on $\{|z|>1\}$, which has no critical points, so the same is true for Q! ∎

Similar analysis can be done for any CER.